

Shortest Paths in Distance-Regular Graphs

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Kernels, Graphs and Shortest Paths

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Abstract

We aim here at introducing a new point of view of the Laplacian of a graph, Γ . With this purpose in mind, we consider \mathcal{L} as a kernel on the finite space $V(\Gamma)$, in the context of the Potential Theory. Then we prove that \mathcal{L} is a nice kernel, since it verifies some fundamental properties as maximum and energy principles and the equilibrium principle on any proper subset of $V(\Gamma)$. If Γ is a proper set of a suitable host graph, then the equilibrium problem for Γ can be solved and the number of the different components of its equilibrium measure leads to a bound on the diameter of Γ . In particular, we obtain the structure of the shortest paths of a distance-regular graph. As a consequence, we find the intersection array in terms of the equilibrium measure. Finally, we give a new characterization of strongly regular graphs.

Key words. Distance-regular graph, shortest path, equilibrium potential, capacity.

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1 Introduction

The discrete Laplacian on a graph and the solution of some associated Dirichlet problems have been widely considered for solving motley problems including, conductance of an electrical network and bounds on the diameter of a graph, among others [1, 4, 5, 8, 9, 11, 14, 15]. The discrete Laplacian on a graph is usually seen as the discrete version of the Laplace operator on Riemannian manifolds and the spectral methods are the main tool.

On the other hand, the quadratic form associated with the Laplace operator, which is closely related to the spectral theory via the Rayleigh quotient, has been also considered to analyze some extremal problems in electrical networks, [10, 16, 17, 18]. These results are obtained when the quadratic form is seen as a Dirichlet form on a Dirichlet space [3]. The elements of this space are potentials with respect to the Green kernel of the Laplace operator. However, as this kernel is formally expressed as a power series, it is difficult to derive properties of the potentials from it, so most properties of the potentials are directly obtained from the Dirichlet forms.

We aim here at introducing another aspect of the relation between Potential and Graph Theories. We consider the discrete Laplacian on a graph as a kernel instead of an operator and we develop the associated Potential Theory. The Laplacian kernel verifies some fundamental principles of the Potential Theory. These principles allow us to obtain information about the connection between a subset of vertices and its complementary as well as about the distance between vertices.

There is no question that the Laplacian of a graph contains information about the connection between vertices. A positive measure on the vertices of a graph, determines a subset of points (its support) as well as a positive weight for each one of them. Therefore, if we choose a uniform measure, for example, the characteristic of a subset of the graph, the potential in each vertex of the subset coincides with its exterior degree. Among all the positive measures with support in a subset of vertices that we can consider, the measure that gives equal potential in each vertex of the subset, must give the maximum information about the exterior connection of the subset. The existence and uniqueness of such a measure, the so-called equilibrium measure of the subset, will be proven in Section 3. As we may expect, this equilibrium measure is uniform if and only if the exterior degree of each vertex is constant.

In addition, the mass of the equilibrium measure, i.e., the Wiener capacity of the subset, provides information not only about the inner connection of the subset, but also about the connection with its complementary. In particular, we prove that the capacity is additive with respect to the connected components of the subgraph induced by a subset.

It is not possible to obtain an equilibrium measure for the whole vertex set of a graph. Therefore, we cannot know connection properties of the whole graph by applying the above mentioned tools directly. To do so, we embed it into a host graph by employing a

commonly used technique in the context of electrical networks (see [3, 6].) It consists of adding a new vertex joined to the graph through a new edge. Although this embedding partly modifies the structure of the initial graph, its equilibrium measure can recognize some properties of the connection between the vertices of the graph. For instance, when this technique is applied to a distance-regular graph, the equilibrium measure recovers its intersection array. Furthermore, in this case the equilibrium measure assigns a mass to each vertex which only depends on its distance to the exterior. This allows to build shortest paths between any pair of vertices. The results related to distance-regular graphs are developed in Section 4, which concludes with a complete characterization of strongly regular graphs.

To sum up, the results here obtained by considering the discrete Laplacian of a graph as a kernel, mainly hinges on the knowledge of the equilibrium measures. Let us point out that the effective computation of such measures can be accomplished using standard techniques of Mathematical Programming (see [2]). Specifically, the computation can be performed in two ways, either by solving a linear mathematical programming problem related to the potentials of the measures or by solving a convex quadratic mathematical programming problem related to the energy of the measures.

Throughout the paper, $\Gamma = (V, E)$ denotes a (simple and finite) connected graph, with vertex set V , $|V| = n$, and edge set E . The distance from x to y is denoted by $d(x, y)$ and $d = d(\Gamma) = \max\{d(x, y) : x, y \in V(\Gamma)\}$ stands for the *diameter* of Γ . Given $x \in \Gamma$, we write as $\Gamma_i(x)$ the set of vertices y such that $d(x, y) = i$. In particular, $\Gamma(x) = \Gamma_1(x)$ denotes the set of vertices adjacent to x . Its cardinal is the *degree* of x , $\delta(x) = |\Gamma(x)|$. A graph is called *k-regular* if each vertex has the same degree k . Given $F \subset V$, $\langle F \rangle$ stands for the induced subgraph. Moreover, we denote by F^c its complementary in V and we consider the subsets $\partial(F) = \{x \in F^c : (x, y) \in E \text{ for some } y \in F\}$ and $\text{Ext}(F) = F^c \setminus \partial(F)$. In addition, for $x \in F$, we call *exterior degree of x with respect to F* the number $\partial^-(x) = |\Gamma(x) \cap F^c|$.

The *Laplacian matrix* of Γ is the $(n \times n)$ -matrix $\mathcal{L} = \mathcal{L}(\Gamma)$ indexed by the vertices of Γ , whose entries \mathcal{L}_{xy} are given by $\mathcal{L}_{xy} = -1$ if x is adjacent to y , ($x \sim y$), $\mathcal{L}_{xx} = \delta(x)$ and $\mathcal{L}_{xy} = 0$ otherwise. The matrix \mathcal{L} is symmetric and positive semidefinite.

2 Some basic concepts of Potential Theory

This section is devoted to introduce the main definitions and results of Potential Theory that we will use later. We only expose those properties which will have repercussions on the rest of the paper. For our purposes, it suffices to consider the potential and the energy of mass distribution on a compact space X with respect to a continuous and symmetric *Kernel*, $\mathcal{K} : X \times X \rightarrow \mathbb{R}$. All results and their proofs can be found in Fuglede [12].

If μ is a positive Radon measure, its *support* and its *mass* will be denoted by $S(\mu)$ and $\|\mu\|$, respectively. For each $F \subset X$ we denote by $\mathcal{M}^+(F)$ the set of positive Radon

measure with support in F and we consider the set $\mathcal{M}^1(F) = \{\mu \in \mathcal{M}^+(F) : \|\mu\| = 1\}$.

Given $\mu \in \mathcal{M}^+(X)$, we call *Potential of μ and Energy of μ* with respect to \mathcal{K} , the function and the value given by

$$U_\mu(x) = \int_X \mathcal{K}(x, y) d\mu(y) \quad \text{and} \quad I(\mu) = \int_X U_\mu(x) d\mu(x),$$

respectively. Also, we consider the functions $U, W : \mathcal{M}^+(X) \rightarrow \mathbb{R}$ given by

$$U(\mu) = \max_{x \in X} U_\mu(x) \quad \text{and} \quad W(\mu) = \max_{x \in S(\mu)} U_\mu(x)$$

and for each $F \subset X$, the values

$$I(F) = \inf_{\mu \in \mathcal{M}^1(F)} I(\mu), \quad U(F) = \inf_{\mu \in \mathcal{M}^1(F)} U(\mu) \quad \text{and} \quad W(F) = \inf_{\mu \in \mathcal{M}^1(F)} W(\mu).$$

When \mathcal{K} satisfies $I(X) \geq 0$, the value

$$\text{cap}(F) = \frac{1}{I(F)},$$

is known as the *Wiener capacity* of F . Note that $\text{cap}(F)$ is strictly positive for all non-empty set $F \subset X$, but it has not to be necessarily finite.

For each compact set $F \subset X$, the following extremal problems are posed:

Find $\sigma, \nu, \lambda \in \mathcal{M}^1(F)$ such that $I(\sigma) = I(F)$, $U(\nu) = U(F)$ and $W(\lambda) = W(F)$.

It is well known that $W(F) = I(F)$ for every compact set $F \subset X$. Moreover, their associated extremal measures are equal and they will be called *capacitary measures* for F . A kernel \mathcal{K} is said to satisfy the *maximum principle* if $U(\mu) = W(\mu)$ for every $\mu \in \mathcal{M}^+(X)$. Hence, if \mathcal{K} verifies the maximum principle, then for every compact set F

$$I(F) = U(F) = W(F)$$

and they have the same extremal measures. In addition, if I is strictly convex on $\mathcal{M}^1(F)$ (i.e., \mathcal{K} verifies the *energy principle* on F), there exists a unique capacitary measure.

For each compact set $F \subset X$, a measure $\mu \in \mathcal{M}^1(F)$ verifies that $I(\mu) = I(F)$ if and only if its potential satisfies the following inequalities

$$\begin{aligned} U_\mu(x) &\geq I(F) && \text{in } F, \\ U_\mu(x) &\leq I(F) && \text{in } S(\mu). \end{aligned} \tag{1}$$

If \mathcal{K} satisfies the maximum principle, then $U_\mu(x) \leq I(F)$ in X and *a fortiori* $U_\mu(x) = I(F)$ in F .

Finally, suppose that \mathcal{K} is a continuous symmetric kernel on X satisfying the maximum principle and $I(X) \geq 0$. If F is a non-empty compact subset of X with finite capacity and \mathcal{K} verifies the energy principle on F , then there exists a unique measure $\nu \in \mathcal{M}^+(F)$ solution of the so-called *equilibrium problem*:

$$\text{Find } \mu \in \mathcal{M}^+(F) \text{ such that } U_\mu(x) = 1 \text{ in } F.$$

Clearly, if σ satisfies $I(\sigma) = I(F) = U(F)$ (i.e., σ is the capacitary measure), then $\nu = \text{cap}(F)\sigma$ and it is called the *equilibrium measure* for F .

3 The Laplacian Kernel

The purpose of this section is to construct a Potential Theory in the context of Graph Theory in such a way that we can apply the results of the previous section.

Let $\Gamma = (V, E)$ be a graph. We consider the vertex set of Γ as the underlying space, i.e., $X = V$. Since V is a finite set, kernels and positive measures on V are identified with symmetric matrices and vectors of the positive cone of \mathbb{R}^n , respectively. So, if $\mu \in \mathcal{M}^+(V)$, then $S(\mu) = \{x \in V : \mu(x) > 0\}$ and $\|\mu\| = \sum_{x \in V} \mu(x)$.

For each $F \subset V$, let $\mathbf{1}_F \in \mathcal{M}^+(F)$ denote the measure given by $\mathbf{1}_F(x) = \begin{cases} 1 & \text{if } x \in F \\ 0 & \text{if } x \notin F \end{cases}$.

If $F = V$ the subscript will be omitted. We say that a measure $\mu \in \mathcal{M}^+(F)$ is a *uniform measure* on F if $\mu = a\mathbf{1}_F$ for some $a > 0$.

If \mathcal{K} is a kernel and $\mu \in \mathcal{M}^+(V)$, then the potential and the energy of μ can be identified with the vector $\mathcal{K}\mu$ and the value $\langle \mathcal{K}\mu, \mu \rangle$ respectively, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in \mathbb{R}^n . Therefore, \mathcal{K} fulfills the energy principle on $F \subset V$ iff \mathcal{K} is strictly positive definite on $\{\mu \in \mathbb{R}^n : \mu(x) = 0 \text{ if } x \notin F \text{ and } \sum_{x \in V} \mu(x) = 0\}$.

The extremal problems described in Section 2 consist of finding $\sigma, \eta \in \mathcal{M}^1(F)$ such that

$$I(\sigma) = \min_{\mu \in \mathcal{M}^1(F)} \langle \mathcal{K}\mu, \mu \rangle \quad \text{and} \quad U(\eta) = \min_{\mu \in \mathcal{M}^1(F)} \max_{x \in V} \mathcal{K}\mu(x).$$

The energy extremal problem becomes a Quadratic Programming Problem, since $\mathcal{M}^1(F)$ is described by means of linear constraints. On the other hand, the potential extremal problem can be re-written as

$$U(\eta) = \min_{\substack{\mu \in \mathcal{M}^1(F) \\ \mathcal{K}\mu \leq u\mathbf{1}}} u \tag{2}$$

Therefore, it becomes a Linear Programming Problem.

In the sequel, we will consider the Laplacian of Γ , \mathcal{L} , as a kernel on V . Then, the potential of $\mu \in \mathcal{M}^+(V)$ is given by

$$\mathcal{L}\mu(x) = \sum_{y \sim x} (\mu(x) - \mu(y)) = \delta(x)\mu(x) - \sum_{y \sim x} \mu(y),$$

and the energy of $\mu \in \mathcal{M}^+(V)$ is the value

$$I(\mu) = \sum_{(x,y) \in E} (\mu(x) - \mu(y))^2.$$

The following results establish that \mathcal{L} is a kernel verifying the maximum and energy principles. This will enable us to tackle the equilibrium problem for all proper subsets $F \subset V$.

Proposition 3.1 *The Laplacian kernel verifies the maximum principle.*

Proof. Given $\mu \in \mathcal{M}^1(F)$, then $\mathcal{L}\mu(x) = 0$ if $x \in \text{Ext}(F)$ and $\mathcal{L}\mu(x) = -\sum_{y \sim x} \mu(y) \leq 0$ if $x \in \partial(F)$. Therefore, it suffices to prove that there exists a vertex $x \in F$ such that $\mathcal{L}\mu(x) \geq 0$. If we chose $x \in F$ such that $\mu(x) = \max_{y \in F} \mu(y)$, then $\mathcal{L}\mu(x) \geq 0$. ■

Proposition 3.2 *For each $F \subset V$, the Laplacian kernel verifies the energy principle on F . Moreover, if F is a proper subset, $I(F) > 0$.*

Proof. Note that $\langle \mathcal{L}\mu, \mu \rangle \geq 0$ and $\langle \mathcal{L}\mu, \mu \rangle = 0$ iff $\mu = a\mathbf{1}$, $a \in \mathbb{R}$. Therefore, \mathcal{L} is strictly positive definite on $\{\mu \in \mathbb{R}^n : \langle \mu, \mathbf{1} \rangle = 0\}$. On the other hand, if F is a proper subset of V , for each $\mu \in \mathcal{M}^1(F)$ there exists $x \in F^c$ such that $\mu(x) = 0$, which implies that $I(\mu) > 0$ and hence, $I(F) > 0$. ■

From the proof of the above proposition, we get that $I(V) = 0$ and its unique capacitary measure is uniform, i.e., $\sigma = \frac{1}{n}\mathbf{1}$.

Corollary 3.3 *For each proper subset $F \subset V$ there exists a unique equilibrium measure ν , i.e.,*

$$\mathcal{L}\nu(x) = 1 \quad \forall x \in F.$$

Moreover, $S(\nu) = F$.

Proof. The first part is a consequence of Propositions 3.1 and 3.2.

Assume that $\nu(x) = 0$ for some $x \in F$. Then $\mathcal{L}\nu(x) = -\sum_{y \sim x} \nu(y) \leq 0$, which contradicts that ν is the equilibrium measure. ■

As pointed out in the continuous case, the equilibrium measure ν is equal to $\text{cap}(F)\sigma$, where σ is the capacitary measure for F . In this case, we have that $(I(F), \sigma)$ is either the solution of the Quadratic Programming Problem related to the energy or the solution of the Linear Programming Problem (2).

Up to now we have shown the basic structure of the Potential Theory on a graph that arises when the Laplacian kernel is considered. Our next goal is to derive properties of the equilibrium measures as well as of the Wiener capacities of subsets. Firstly, we characterize the conditions for an equilibrium measure to be uniform.

Proposition 3.4 *Let $F \subset V$ be a proper subset. Then, its equilibrium measure is uniform iff $\partial^-(x) = \partial^-(y)$ for all $x, y \in F$.*

Proof. It suffices to observe that $\mathcal{L}\mathbf{1}_F(x) = \partial^-(x)$ for all $x \in F$. ■

The following proposition will be useful for later results.

Proposition 3.5 *Let F be a proper subset of V and ν its equilibrium measure. Then*

$$|F| = \sum_{x \in F} \partial^-(x)\nu(x).$$

Proof. As ν is the equilibrium measure, then $\mathcal{L}\nu(x) = 1$, for all $x \in F$ and hence,

$$\begin{aligned} |F| &= \sum_{x \in F} \mathcal{L}\nu(x) = \sum_{x \in F} \sum_{y \sim x} (\nu(x) - \nu(y)) \\ &= \sum_{x \in F} \sum_{\substack{y \sim x \\ y \in F}} (\nu(x) - \nu(y)) + \sum_{x \in F} \sum_{\substack{y \sim x \\ y \notin F}} \nu(x) = \sum_{x \in F} \partial^-(x)\nu(x). \quad \blacksquare \end{aligned}$$

Proposition 3.6 *Let $F \subset V$ such that $F = \bigcup_{i=1}^s F_i$, where $F_i, i = 1, \dots, s$ are the vertex sets of the connected components of $\langle F \rangle$. Then*

$$\text{cap}(F) = \sum_{i=1}^s \text{cap}(F_i).$$

Proof. If $F = V$, then $s = 1$, because Γ is connected. Hence, the result holds.

Suppose that F is a proper subset of V . For each $i = 1, \dots, s$, let μ^i be the capacitary measure for F_i . If $\beta = \sum_{i=1}^s \text{cap}(F_i)$ and we consider $\mu \in \mathcal{M}^1(F)$ defined by

$$\mu = \frac{1}{\beta} \sum_{i=1}^s \text{cap}(F_i) \mu^i,$$

then $\mathcal{L}\mu = \frac{1}{\beta} \sum_{i=1}^s \text{cap}(F_i) \mathcal{L}\mu^i$. If $x \in F$, there exists k such that $x \in F_k$. Moreover $x \in \text{Ext}(F_j)$, for all $j \neq k$. Then,

$$\mathcal{L}\mu^i(x) = \begin{cases} I(F_k) & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases}$$

Hence, $\mathcal{L}\mu(x) = \frac{1}{\beta}$ for all $x \in F$, which implies that $I(F) = \frac{1}{\beta}$ and the result follows. ■

Corollary 3.7 *Let $F = \{x_1, \dots, x_s\}$ be a set of independent vertices. Then,*

$$\text{cap}(F) = \sum_{i=1}^s \frac{1}{\delta(x_i)}.$$

Proof. If $\mu = \mathbf{1}_{x_i}$, the Dirac measure on x_i , then $\mathcal{L}\mu(x_i) = \delta(x_i)$. Therefore, $I(x_i) = \delta(x_i)$. The result follows by applying the previous proposition. ■

The result of Proposition 3.6 states that the capacity is additive with respect to the connected components of a induced subgraph. However, it is not true for an arbitrary union of subsets of V as the following example shows.

Lemma 3.8 *Let $F = \{x, y\}$ such that $(x, y) \in E$. Then*

$$\text{cap}(F) > \text{cap}(x) + \text{cap}(y).$$

Proof. The capacitary measure for F , σ , must satisfy $\delta(x)\sigma(x) - \sigma(y) = \delta(y)\sigma(y) - \sigma(x)$ and $\sigma(x) + \sigma(y) = 1$. Then, $\sigma(x) = \frac{\delta(y) + 1}{\delta(x) + \delta(y) + 2}$ and $\sigma(y) = \frac{\delta(x) + 1}{\delta(x) + \delta(y) + 2}$. Therefore,

$$\text{cap}(F) = \frac{\delta(x) + \delta(y) + 2}{\delta(x)\delta(y) - 1} > \frac{1}{\delta(x)} + \frac{1}{\delta(y)}. \quad \blacksquare$$

The above lemma says that the Wiener capacity for the Laplacian kernel, is not sub-additive. This is due to the fact that the Laplacian is not a positive matrix. However, this example verifies the following equality:

$$\frac{1}{I(\{x, y\}) + 1} = \frac{1}{I(x) + 1} + \frac{1}{I(y) + 1}.$$

If $F \subset V$, the value $(I(F) + 1)^{-1}$ can be seen as the Wiener capacity of F with respect to the positive kernel $\mathcal{L} + J$, where J denotes the matrix whose entries are equal to one. In fact, the Wiener capacity is subadditive for a positive kernel, (see [12, p. 157].) In particular, we have the following result.

Proposition 3.9 *Let $F_1, \dots, F_s \subset V$ and $F = \bigcup_{i=1}^s F_i$. Then,*

$$(I(F) + 1)^{-1} \leq \sum_{i=1}^s (I(F_i) + 1)^{-1}.$$

Corollary 3.10 *Let $F \subset V$ be a proper subset. Then, $\text{cap}(F)\text{cap}(F^c) \geq 1$.*

Before ending this section let us determine the Wiener capacities and the capacity measures for proper subsets of some nice graphs which help us to study the sharpness of the lower bound in the above corollary. Table 1 shows such capacities and measures for connected subsets $F = \{x_1, \dots, x_s\}$ of cardinal $s < n$.

Notice that the product $\text{cap}(F)\text{cap}(F^c)$ can be much larger than one. For instance in a cycle $\text{cap}(F)\text{cap}(F^c) \geq \frac{n(n^2-1)}{24}$. However, in a complete graph this product is equal to one. The differences in the behaviour of the capacity products are due to the different degrees of connection between the vertices of F and F^c . In particular, the following result characterizes when equality holds in Corollary 3.10.

Proposition 3.11 *Let $F \subset V$ be a proper subset. Then,*

$$\text{cap}(F)\text{cap}(F^c) = 1 \iff \max\{d(x, y) : x \in F, y \in F^c\} = 1.$$

Moreover, the capacity measures for F and F^c are the uniform measures on F and F^c respectively.

Proof. Note that $\max\{d(x, y) : x \in F, y \in F^c\} = 1$ iff $\partial^-(x) = |F^c|$ for all $x \in F$ and $\partial^-(y) = |F|$ for all $y \in F^c$. In addition, the uniform measures on F and F^c , $\mu_1 = \frac{1}{|F|}\mathbf{1}_F$ and $\mu_2 = \frac{1}{|F^c|}\mathbf{1}_{F^c}$, satisfy $\mathcal{L}\mu_1 = \frac{|F^c|}{|F|}$ on F and $\mathcal{L}\mu_2 = \frac{|F|}{|F^c|}$ on F^c , respectively. Therefore, they are the capacity measures for F and F^c and $\text{cap}(F)\text{cap}(F^c) = 1$.

Γ	F	$\sigma(x_i)$	$\text{cap}(F)$
complete graph, K_n		$\frac{1}{s}$	$\frac{s}{n-s}$
bipartite complete graph, $K_{p,q}$	$ F \cap V_0 = s_0$ $ F \cap V_1 = s_1$	$\frac{p+s_1}{ps_0+qs_1+2s_0s_1}$ $\frac{q+s_0}{ps_0+qs_1+2s_0s_1}$	$\frac{ps_0+qs_1+2s_0s_1}{pq-s_0s_1}$
path, P_n	$\delta(x_s) = 1$	$3\frac{2is-i(i-1)}{s(s+1)(2s+1)}$	$\frac{1}{6}s(s+1)(2s+1)$
path, P_n or cycle, C_n	$\delta(x_i) = 2$	$i = 1, \dots, \lceil \frac{s}{2} \rceil$ $6\frac{is-i(i-1)}{s(s+1)(s+2)}$ $\sigma(x_{s-i+1}) = \sigma(x_i)$	$\frac{1}{12}s(s+1)(s+2)$

Table 1: Some capacities and capacity measures.

Conversely, if $\mathcal{K} = \mathcal{L} + J$ and we consider $\mathbf{1} = \mathbf{1}_F + \mathbf{1}_{F^c}$, then

$$\begin{aligned}
n^2 &= \langle \mathcal{K}\mathbf{1}, \mathbf{1} \rangle = \langle \mathcal{K}\mathbf{1}_F, \mathbf{1}_F \rangle + \langle \mathcal{K}\mathbf{1}_{F^c}, \mathbf{1}_{F^c} \rangle + 2\langle \mathcal{K}\mathbf{1}_F, \mathbf{1}_{F^c} \rangle \\
&\geq (I(\mathbf{1}_F) + |F|^2) + (I(\mathbf{1}_{F^c}) + |F^c|^2) \geq |F|^2(I(F) + 1) + |F^c|^2(I(F^c) + 1) \\
&\geq \frac{(|F| + |F^c|)^2}{\frac{1}{I(F)+1} + \frac{1}{I(F^c)+1}} = \frac{n^2}{\frac{1}{I(F)+1} + \frac{1}{I(F^c)+1}},
\end{aligned}$$

where the last inequality has been obtained by applying the Cauchy-Schwarz inequality in the form: $(a+b)^2 \leq (c+d)\left(\left(\frac{a}{\sqrt{c}}\right)^2 + \left(\frac{b}{\sqrt{d}}\right)^2\right)$.

On the other hand, $\text{cap}(F)\text{cap}(F^c) = 1$ iff $\frac{1}{I(F)+1} + \frac{1}{I(F^c)+1} = 1$. Therefore, by using

the above inequalities, we conclude that

$$\text{cap}(F)\text{cap}(F^c) = 1 \implies \langle \mathcal{K}\mathbf{1}_F, \mathbf{1}_{F^c} \rangle = 0.$$

Finally, it is enough to observe that

$$\max\{d(x, y) : x \in F, y \in F^c\} = 1 \text{ iff } \langle \mathcal{K}\mathbf{1}_F, \mathbf{1}_{F^c} \rangle = 0,$$

since

$$|F||F^c| - \sum_{y \in F^c} \partial^-(y) = \langle \mathcal{K}\mathbf{1}_F, \mathbf{1}_{F^c} \rangle = \langle \mathcal{K}\mathbf{1}_{F^c}, \mathbf{1}_F \rangle = |F||F^c| - \sum_{x \in F} \partial^-(x). \quad \blacksquare$$

Up to now, we have analyzed the equilibrium problem and some of its properties for any proper subset of the vertex set of a graph. Clearly, the equilibrium problem for the vertex set of a graph, with respect to the Laplacian kernel, could not be solved unless we embed it in a host graph. Although there exist motley ways of do this, we will proceed in such a way that the influence of the joined elements to the initial graph is minimum and the information about the inner connection of V remains as much as possible. For instance, if we consider a unique new vertex joined with each of the vertices of the graph through a new edge, by Proposition 3.4 we conclude that the equilibrium measure is the uniform measure on V and the capacity of V is $\frac{1}{n}$. Therefore, this embedding does not give us more information about the inner connection of V . For our purposes, it will be better to add to the graph a unique vertex and a unique edge.

Specifically, let $\Gamma = (V, E)$ be a graph and consider a new vertex x_{n+1} , which will be joined through an edge to a fix vertex $x \in V$. Let $\Gamma^x = (V^x, E^x)$, where $V^x = V \cup \{x_{n+1}\}$ and $E^x = E \cup (x, x_{n+1})$. We will call this graph *the extended graph of Γ with respect to x* . Let \mathcal{L}^x the Laplacian of Γ^x . In this case, the submatrix $\mathcal{L}_{|\Gamma}^x$ coincides with \mathcal{L} except in its diagonal element $(\mathcal{L}^x)_{xx}$ which is equal to $\delta(x) + 1$.

Let us consider now the equilibrium problem for V as a proper set of V^x . Then, using the previous results, there exists a unique equilibrium measure, ν^x , for V . Therefore, the potential of ν^x satisfies

$$\begin{aligned} \mathcal{L}^x \nu^x(y) &= 1 && \text{if } y \in V \\ \mathcal{L}^x \nu^x(y) &= -\nu^x(x) && \text{if } y = x_{n+1}. \end{aligned} \quad (3)$$

In the sequel, we call *equilibrium array for $V \subset V^x$* to $q(x) = \{q_0(x), q_1(x), \dots, q_t(x)\}$, the set of different components of ν^x , where it is assumed that $q_0(x) < q_1(x) < \dots < q_t(x)$. Note that the *length* of the equilibrium array, $t + 1$, is larger than one, unless $n = 1$, because ν^x cannot be uniform. On the other hand, $q_0(x)$ only depends on the order of Γ , since $q_0(x) = n$ by Proposition 3.5. Also, we will consider the positive integers

$m_i(x) = |\{y \in V : \nu^x(y) = q_i(x)\}|$, $i = 0, \dots, t$, i.e., the multiplicity of each element of the equilibrium array.

The equilibrium measure enables us to obtain an upper bound on the distance between vertices.

Proposition 3.12 *Let $V \subset V^x$, ν^x the equilibrium measure and $q(x)$ the equilibrium array for V . Then,*

$$\nu^x(y) = q_i(x) \implies d(y, x) \leq i.$$

In particular, $m_0(x) = 1$.

Proof. We prove the result by mathematical induction.

If $\nu^x(y) = q_0(x)$, then $y = x$, otherwise $\Gamma(y) \subset V$ and $\nu^x(y) \leq \nu^x(z)$ for all $z \in \Gamma(y)$. Therefore, $\mathcal{L}^x \nu^x(y) = \sum_{z \in \Gamma(y)} (\nu^x(y) - \nu^x(z)) = \sum_{z \in \Gamma(y)} (q_0(x) - \nu^x(z)) \leq 0$, a contradiction since ν is the equilibrium measure. This reasoning also proves that $m_0(x) = 1$.

Suppose that $\nu^x(y) = q_j(x) \implies d(y, x) \leq j$, for all $j = 0, \dots, i$, and let $y \in V$ such that $\nu^x(y) = q_{i+1}(x)$. Assume that for each $z \in \Gamma(y)$ there exists $j \geq i + 1$ such that $\nu^x(z) = q_j(x)$. Then

$$\mathcal{L}^x \nu^x(y) = \sum_{z \sim y} (\nu^x(y) - \nu^x(z)) \leq \sum_{z \sim y} (q_{i+1}(x) - q_{i+1}(x)) = 0,$$

a contradiction again. Therefore, there must exist a vertex $z \in \Gamma(y)$ such that $\nu^x(z) = q_j(x)$, for some $j \leq i$. Then, by using the hypothesis of induction, $d(z, x) \leq i$, which implies $d(y, x) \leq i + 1$. ■

For each vertex $x \in V$ we can consider the equilibrium problem for $V \subset V^x$. Then applying the above proposition, we obtain an upper bound of the distance between any pair of vertices. In particular, the maximum length of all equilibrium arrays minus one is an upper bound of the diameter of the graph.

4 Distance-regular graphs

In this section, we study the case of distance-regular graphs. It seems natural to ask ourselves whether the result of Proposition 3.12 can be improved with additional information about the structure of the considered graphs. In particular, this is the case when the graph is a distance-regular graph. Thus, in this section, we elaborate upon the previous work to derive some new results for such a case. Specifically, we use the equilibrium theory to

determine the distance between any pair of vertices as well as a shortest path between them.

A connected graph Γ is called *distance-regular* if there are integers $b_i, c_i, i = 0, \dots, d$ such that for any two vertices $x, y \in \Gamma$ at distance $i = d(x, y)$, there are exactly c_i neighbours of y in $\Gamma_{i-1}(x)$ and b_i neighbours of y in $\Gamma_{i+1}(x)$. In particular, Γ is regular of degree $k = b_0$.

The sequence

$$\iota(\gamma) = \{b_0, b_1, \dots, b_{d-1}; c_1, \dots, c_d\},$$

is called the *intersection array* of Γ . In addition, $a_i = k - c_i - b_i$ is the number of neighbours of y in $\Gamma_i(x)$, for $d(x, y) = i$. Clearly, $b_d = c_0 = 0, c_1 = 1$ and the diameter of Γ is d . Moreover, $1 \leq c_2 \leq \dots \leq c_d$.

For any vertex $x \in \Gamma$ the number of vertices at distance i from it, i.e., $|\Gamma_i(x)|$, will be denoted by k_i . This number does not depend on the vertex, x , and the following equalities hold:

$$k_0 = 1, k_1 = k, k_{i+1}c_{i+1} = k_i b_i, i = 2, \dots, d-1. \quad (4)$$

For basic concepts and properties on distance-regular graphs, we refer to the reader to Brouwer, Cohen and Neumaier [7].

As usual, we consider the *distribution diagram* associated with the intersection array of the graph. Then, the Laplacian matrix can be represented by the following tridiagonal $(d+1) \times (d+1)$ matrix:

$$\mathcal{L}_D = \begin{pmatrix} k & -b_0 & 0 & \dots & 0 & 0 \\ -c_1 & k - a_1 & -b_1 & \dots & 0 & 0 \\ 0 & -c_2 & k - a_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & k - a_{d-1} & -b_{d-1} \\ 0 & 0 & 0 & \dots & -c_d & k - a_d \end{pmatrix}$$

Most of the results about distance-regular graphs are obtained by using the matrix \mathcal{L}_D . This will also happen in our development. For our purposes we have to consider Γ^x , the extended graph with respect to x , and the matrix \mathcal{L}_D^x which is equal to \mathcal{L}_D except for the first diagonal entry $(\mathcal{L}_D^x)_{11} = k + 1$. We will prove that the system $\mathcal{L}_D^x \lambda = 1$ has the equilibrium array of the equilibrium problem for V with respect to the kernel \mathcal{L}^x as unique solution.

Now, we tackle the existence of a solution of the system $\mathcal{L}_D^x \lambda = 1$, i.e., the system:

$$\begin{cases} (k+1)\lambda_0 - k\lambda_1 = 1 \\ -c_i\lambda_{i-1} + (b_i + c_i)\lambda_i - b_i\lambda_{i+1} = 1, & i = 1, \dots, d-1 \\ -c_d\lambda_{d-1} + (b_d + c_d)\lambda_d = 1. \end{cases} \quad (5)$$

In what follows let $\gamma_i = \lambda_{i+1} - \lambda_i$, $i = 0, \dots, d-1$ and $\gamma_{-1} = \lambda_0$. Then $\lambda = (\lambda_0, \dots, \lambda_d)$ is a solution of the system (5) iff $\gamma = (\gamma_{-1}, \dots, \gamma_{d-1})$ is a solution of the system:

$$\begin{cases} \gamma_{-1} - k\gamma_0 = 1 \\ c_i\gamma_{i-1} - b_i\gamma_i = 1, & i = 1, \dots, d-1 \\ c_d\gamma_{d-1} = 1 \end{cases} \quad (6)$$

Proposition 4.1 *Let $\Gamma = (V, E)$ be a distance-regular graph. Then the system (6) has a unique solution given by:*

$$\gamma_{-1} = n, \quad \gamma_i = \frac{1}{c_{i+1}k_{i+1}} \left(n - \sum_{j=0}^i k_j \right), \quad i = 0, \dots, d-1.$$

In addition, $\gamma_i > 0$ for all $i = 0, \dots, d-1$.

Proof. The proof is by mathematical induction on $i = d-1, \dots, 0$. For $i = d-1$ the result follows immediately from the last equation of the system. Suppose now that it is verified for i , then

$$c_i\gamma_{i-1} = 1 + b_i\gamma_i = 1 + \frac{b_i}{k_{i+1}c_{i+1}} \left(n - \sum_{j=0}^i k_j \right) = \frac{1}{k_i} \left(k_i + n - \sum_{j=0}^i k_j \right) = \frac{1}{k_i} \left(n - \sum_{j=0}^{i-1} k_j \right).$$

Finally, $\gamma_{-1} = n$. In addition, $\gamma_i > 0$ for all $i = 0, \dots, d-1$, since $n = \sum_{j=0}^d k_j$. ■

Corollary 4.2 *The system $\mathcal{L}_D^x \lambda = 1$ has a unique solution such that*

$$n = \lambda_0 < \lambda_1 < \dots < \lambda_d.$$

Proposition 4.3 *Let $\Gamma = (V, E)$ be a distance-regular graph, λ the solution of $\mathcal{L}_D^x \lambda = 1$ and $q(x)$ the equilibrium array for $V \subset V^x$. Then $\lambda = q(x)$.*

Proof. Let ν^x such that $\nu^x(y) = \lambda_j$ if $d(x, y) = j$. Then

$$\mathcal{L}^x \nu^x(y) = k\lambda_j - c_j\lambda_{j-1} - a_j\lambda_j - b_j\lambda_{j+1} = (\mathcal{L}_D^x \lambda)_j = 1.$$

Therefore, ν^x is the equilibrium measure. Furthermore, the equilibrium array of the equilibrium measure is the solution of the system $\mathcal{L}_D^x \lambda = 1$, because the equilibrium measure is unique. ■

Corollary 4.4 *Let $\Gamma = (V, E)$ be a distance-regular graph. Then the equilibrium measure, ν^x , for $V \subset V^x$ verifies*

$$\nu^x(y) = q_i(x) \iff d(x, y) = i.$$

Moreover, $q(x)$ is independent of x .

Based on the above corollary we call equilibrium array for V to $q = q(x)$ for any $x \in V$. Note that the equilibrium measure does depend on x , because its mass on $y \in V$ depends on the distance between x and y .

A straightforward consequence of the above corollary is that the diameter of a distance-regular graph is equal to the length of its equilibrium array minus one. Furthermore, we can obtain a shortest path between any pair of vertices. Namely, given $x, y \in V$ we solve the Linear Programming Problem (2) with respect to the extended laplacian kernel, \mathcal{L}^x , to obtain the equilibrium measure ν^x . Then we find i such that $\nu^x(y) = q_i$. The next step consists of applying the Shortest Path Algorithm described in Figure 1 to find the path $w_0 = x, w_1, \dots, w_{i-1}, w_i = y$.

We have shown the equivalence between the solution of the system $\mathcal{L}_D^x \lambda = 1$ and the solution of the equilibrium problem for $V \subset V^x$. This has enabled us to determine the equilibrium measure from the intersection array. The following result shows that the equilibrium measure also determines the intersection array of a distance-regular graph.

Proposition 4.5 *Let Γ be a distance-regular graph and q its equilibrium array. Then $d(\Gamma) = d$ and*

$$k_i = m_i, \quad b_i = \frac{1}{m_i(q_{i+1} - q_i)} \sum_{j=i+1}^d m_j, \quad c_{i+1} = \frac{1}{m_{i+1}(q_{i+1} - q_i)} \sum_{j=i+1}^d m_j, \quad i = 0, \dots, d-1.$$

Proof. The proof is straightforward using that q is the unique solution of $\mathcal{L}_d^x \lambda = 1$ and applying Proposition 4.1. ■

An application of the equilibrium problem refers to the estimation of the effective resistance of a resistive electrical network. If the underlying graph is a distance-regular graph and q is its equilibrium array, the effective resistance between two vertices $x, y \in V$ at distance i is given by $r_i = r_{xy} = \frac{2}{n}(q_i - q_0)$. This is because that the equilibrium array is the solution of the system (5), which is equivalent to the system solved by Biggs [4, Theorem C] to determinate the effective resistance.

We finish this paper showing that the equilibrium measures, ν^x , $x \in V$, characterize the *strongly regular graphs*, i.e., distance-regular graphs with diameter equal to two. If Γ is a strongly regular graph of order n and degree k , any pair of adjacent vertices

have a_1 common neighbours and any two distinct non-adjacent vertices have c_2 common neighbours. It is known that a regular graph Γ is strongly regular iff it has exactly three different eigenvalues (see [13, p. 179]). We obtain an analogous result based on the length of the equilibrium arrays for V .

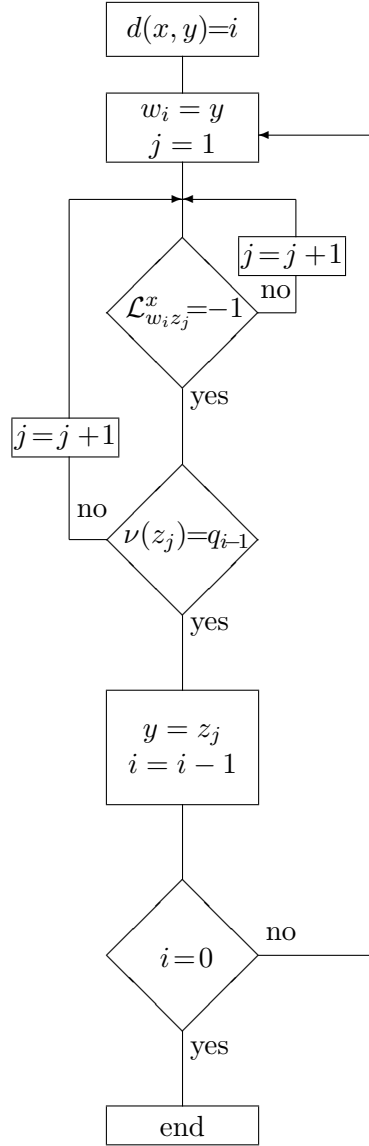


Figure 1: Shortest Path Algorithm.

Theorem 4.6 *Let $\Gamma = (V, E)$ be a k -regular graph. Then, Γ is strongly regular iff for each $x \in V$, $q(x)$ has length equal to three.*

Proof. If Γ is strongly regular, for each $x \in V$, $q(x)$ is independent of x and has length equal to three, because Γ is distance-regular and $d(\Gamma) = 2$.

Conversely, let $x \in V$ and $q(x) = \{q_0(x), q_1(x), q_2(x)\}$ the equilibrium array for $V \subset V^x$, with multiplicities $1 = m_0(x), m_1(x), m_2(x)$.

The first step of the proof consists of showing that

$$\nu^x(y) = q_i(x) \iff d(x, y) = i, \quad i = 1, 2.$$

From Proposition 3.12 we know that if $\nu^x(y) = q_1(x)$, then $y \sim x$. Suppose that there exists a vertex $y \sim x$ such that $\nu^x(y) = q_2(x)$ and take a vertex $z \sim x$ such that $\nu^x(z) = q_1(x)$. Then the potential at y is

$$\mathcal{L}^x \nu^x(y) = kq_2(x) - \alpha q_2(x) - \beta q_1(x) - q_0(x) = 1,$$

where α and β are the number of neighbours of y which have measure $q_2(x)$ and $q_1(x)$ respectively.

Analogously,

$$\mathcal{L}^x \nu^x(z) = kq_1(x) - \alpha' q_2(x) - \beta' q_1(x) - q_0(x) = 1,$$

where α' and β' are defined in a similar way. Subtracting the two last equations and keeping in mind that $k = \alpha + \beta + 1 = \alpha' + \beta' + 1$, we have $(\beta + \alpha' + 1)(q_2(x) - q_1(x)) = 0$ which is a contradiction, since $\beta \geq 0$, $\alpha' \geq 0$ and $q_1(x) < q_2(x)$.

As a result of the case $i = 1$, we also obtain that $\nu^x(y) = q_2(x) \iff d(x, y) = 2$. Therefore, $d(\Gamma) = 2$, $m_1(x) = k$ and $m_2(x) = n - k - 1$.

Now, take $y \in \Gamma(x)$, $a_1^y(x) = |\Gamma(y) \cap \Gamma(x)|$ and $b_1^y(x) = |\Gamma(y) \cap \Gamma_2(x)|$. We show that these numbers are independent of y . Let $y, z \in \Gamma(x)$ and consider the potential at them.

$$\begin{aligned} \mathcal{L}^x \nu^x(y) &= kq_1(x) - a_1^y(x)q_1(x) - b_1^y(x)q_2(x) - \nu_0 = 1 \\ \mathcal{L}^x \nu^x(z) &= kq_1(x) - a_1^z(x)q_1(x) - b_1^z(x)q_2(x) - \nu_0 = 1. \end{aligned}$$

Subtracting these two equations we obtain $(b_1^y(x) - b_1^z(x))(q_1(x) - q_2(x)) = 0$. Therefore, $b_1^y(x) = b_1^z(x)$, since $q_1(x) < q_2(x)$. Keeping in mind that $a_1^y(x) + b_1^y(x) + 1 = k$ for all $y \sim x$, we get that $a_1^y(x) = a_1^z(x)$.

Analogously, let $y \in \Gamma_2(x)$, $a_2^y(x) = |\Gamma(y) \cap \Gamma_2(x)|$ and $c_2^y(x) = |\Gamma(y) \cap \Gamma_1(x)|$. We conclude that these numbers are independent of y by considering the potential at $y, z \in \Gamma_2(x)$ and reasoning as above.

Therefore, for each $x \in V$ we have the array $\{a_1(x), a_2(x), b_1(x), c_2(x)\}$. We finish the proof by showing that these numbers do not depend on x , i.e., $\iota(\Gamma) = \{k, b_1; 1, c_2\}$ is the intersection array of Γ . It suffices to prove that one of the elements of this array

is independent of x , because $a_1(x) = k - b_1(x) - 1$, $kb_1(x) = (n - k - 1)c_2(x)$ and $a_2(x) = k - c_2(x)$.

Let $x, y \in V$ be. If $d(x, y) = 1$, then $a_1(x) = |\Gamma(x) \cap \Gamma(y)| = a_1(y)$. On the other hand, if $d(x, y) = 2$ and z is an adjacent vertex to x and y , then $a_1(x) = |\Gamma(z) \cap \Gamma(x)| = a_1(z) = |\Gamma(z) \cap \Gamma(y)| = a_1(y)$. ■

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